Machinery & Energetics. Journal of Rural Production Research. Kyiv. Ukraine. 2021, Vol. 12, No 3, 107-111
ISSN 2663-1334 (print), ISSN 2663-1342 (online), www.journals.nubip.edu.ua/index.php/Tekhnica

http://dx.doi.org/10.31548/machenergy2021.03.107

УДК 539.3

ON SOME ASPECTS OF IMPLEMENTATION OF BOUNDARY ELEMENTS METHOD IN PLATE THEORY

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Speciality of article: 131 – applied mechanics.

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Article history: Received – April 2021, Accepted – August 2021, Published – 30 September 2021.
Bibl. 7, fig. 5, tabl. 2.

Abstract. The article deals with the problem arising in the construction of a numerical scheme of the first-order boundary element method for plate theory. During construction of such a scheme, the initially smooth boundary of the plate is replaced by a polygonal chain. Due to this replacement the deviation of the numerical results from the actual distribution of deflections and other characteristics is arisen. The reason for this deviation lies in the so-called Sapondzyan's paradox. According to it, the deflection of a plate bounded by a regular polygon does not converge to the deflection of a circular plate with increasing of the polygon sides number. In the paper, on the basis of an analytical consideration of Sapondzyan's problem, the components of the numerical scheme of the boundary element method, which are responsible for the mentioned deviation, are pointed out. The modification of the boundary element method scheme that allows to eliminate given problem is presented. This approach is tested on the example of solving two pairs of problems for elliptical and rectangular plates. The results of numerical solution of those problems confirmed the adequacy of the proposed modification.

Key words: plate theory, deflection, boundary elements method, Sapondzyan`s paradox.

Introduction

It should be noted that in modern mechanics, there are many methods for investigating the propagation of various types of waves in mechanical systems. The study of a one-dimensional and quasi-one-dimensional wave propagation problem in beams and plates led to the development of a number of analytical methods, among which we note the transfer matrix method [7]. This method consists in constructing a matrix, which binds the dynamic and kinematic characteristics. These characteristics are used for the definition the constants of the wave propagation through the eigenvalues of this matrix. Another approach is variational method [4], which is based on energy balance ratios and they are the development of Rayleigh and Rayleigh - Ritz methods. Also, in the vast majority of cases the finite elements method is used for numerical calculations [5], [6]. However, the finite elements method can not be considered as the best complement to analytical methods in investigating of the phenomena of wave propagations, in particular, in periodic systems. Since the corresponding problems are linear boundary problems for the differential equations (ordinary or partial derivatives), the most appropriate method for these purposes is the boundary elements method (BEM). But some nuance can be occurred when boundary elements method is used for solving plate theory problems.

A few decades ago one phenomenon was discovered in the plate theory. Later this phenomenon became known as the paradox by Sapondzyan. It was sufficiently well covered in the literature [1].

In this paper we consider static (or classical) case of corresponding problem. The treatment of this problem for dynamical case has not any principal distinction.

Formulation of problem

Let us consider the essence of this phenomenon. To do this, consider the displacements of two simply supported plates, which are loaded with a constant bending moment on their borders. The plates have a difference shape. The first plate is a circular plate of the unit radius. The second plate is bounded by the regular N-polygon inscribed in the unit circle.

At first glance, it is logical to assume that due passage to the limit $N \to \infty$, the displacement of these plates must be coincide. However, calculations show that the displacements of the N-polygon plate is more small than the displacements of the round one. This fact is due to the fact that the simple supported of plates in the angular points give greater rigidity to the plate. As a result, the contribution of each angular point is arbitrarily small, but due to an increasing of their number, the total contribution aspires to a certain value.

Purpose of research

Therefore, the paradox by Sapondzyan must be taken into account when a numerical scheme of the boundary elements method with linear (the first order) elements is constructing. Since in such scheme the boundary is approximated by polygon, the Sapondzyan's paradox can lead to a violation of one of the postulates of the BEM: when the number of boundary elements is increasing the accuracy of the solution should be increasing too. To clarify this question, let us consider the above formulated problems.

Research results

As is known, in the case of a smooth boundary of the plate, the boundary equations system has the form (see for example [2] or [3]):

$$w(\vec\xi)=\int_\Gamma\Big(-V^*(\vec x,\vec\xi)w(\vec x)-M^*(\vec x,\vec\xi)\theta(\vec x)+$$

$$+\theta^*(\vec x,\vec\xi)M(\vec x)-w^*(\vec x,\vec\xi)V(\vec x)\Big)d\Gamma(\vec x)$$

$$\theta(\vec\xi)=\int_\Gamma\Bigg(-\frac{\partial V^*(\vec x,\vec\xi)}{\partial\vec n_\xi}w(\vec x)-\frac{\partial M^*(\vec x,\vec\xi)}{\partial\vec n_\xi}\theta(\vec x)+$$

$$+\frac{\partial\theta^*(\vec x,\vec\xi)}{\partial\vec n_\xi}M(\vec x)-\frac{\partial w^*(\vec x,\vec\xi)}{\partial\vec n_\xi}V(\vec x)\Bigg)d\Gamma(\vec x), \qquad (1)$$

where $\vec x$ and $\vec\xi$ — points that are lies on the plate boundary (on contour Γ); w — the deflection of the plate; θ — angle of slope of the plate; M and V — bending moment and efficient shear force in points of the plate. After solving system (1) its first relations can be used for determination of deflection within plate. In this case $\vec\xi$ is inner point of plate.

It has been known (see for example [2] or [3]) that the fundamental solution of the equation for the plate bending has the form:

$$w^*(\vec x,\vec\xi)=\frac{r^2}{8\pi D}\ln r\,, \qquad (2)$$

where $r=|\vec x-\vec\xi|$, $\vec x=(x,y)$, $\vec\xi(\xi,\eta)$.

Let us consider an axisymmetric problem for a circular plate of an unit radius (see Fig.1). This is a case where $\Gamma=S_1$ is unit circle, and w, θ, M, V are independent of $\vec x$.

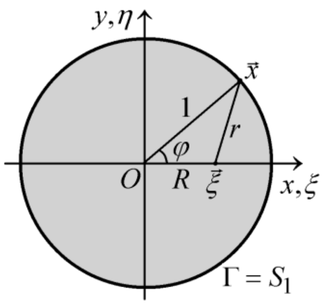

Fig. 1. The geometry of circular plate.

Without loss generality, we will accept that $\vec\xi=(R,0)$. In this case the system of equations (1) can be presented in the following form:

$$w(R)=A(R)\overline{w}+B(R)\overline{\theta}+C(R)\overline{M}+D(R)\overline{V}\,,$$

$$\theta(R)=E(R)\overline{w}+F(R)\overline{\theta}+G(R)\overline{M}+H(R)\overline{V}\,, \qquad (3)$$

where

$$\overline{w}=w(1),\quad \overline{\theta}=\theta(1),\quad \overline{M}=M(1),\quad \overline{V}=V(1)$$

boundary values of corresponding characteristics. After simple but cumbersome transformations the coefficients of system (3) can be founded in the closed form:

$$A(R)=-\int_{S_1}V^*(\vec x,\vec\xi)dS_1(\vec x)=\cdot$$

$$=1+\frac{R}{2\pi}\int_0^{2\pi}\frac{\cos\varphi-R}{1+R^2-2R\cos\varphi}d\varphi=1,$$

$$B(R)=-\int_{S_1}M^*(\vec x,\vec\xi)dS_1(\vec x)=$$

$$=\frac{1}{4\pi}+\frac{1+\nu}{2\pi}\int_0^{2\pi}\Big[1+\ln\Big(1+R^2-2R\cos\varphi\Big)\Big]d\varphi+$$

$$+\frac{1+\nu}{4\pi}R^2\int_0^{2\pi}\frac{\sin^2\varphi\,d\varphi}{1+R^2-2R\cos\varphi}=\frac{1}{4}\Big[(3+\nu)-(1-\nu)R^2\Big],$$

$$C(R)=\int_{S_1}\theta^*(\vec x,\vec\xi)dS_1(\vec x)=$$

$$=\frac{R}{8\pi T}\int_0^{2\pi}\Big[1+\ln\Big(1+R^2-2R\cos\varphi\Big)\Big]d\varphi=-\frac{1+R^2}{4T},$$

$$D(R)=\int_{S_1}w^*(\vec x,\vec\xi)dS_1(\vec x)=$$

$$=\frac{R}{16\pi T}\int_0^{2\pi}\Big[\Big(1+R^2-2R\cos\varphi\Big)\times$$

$$\times\ln\Big(1+R^2-2R\cos\varphi\Big)\Big]d\varphi=\frac{R^2}{4T},$$

$$E(R)=\int_{S_1}\frac{\partial V^*(\vec x,\vec\xi)}{\partial\vec n_\xi}dS_1(\vec x)=-\frac{dA(R)}{dR}=0,$$

$$F(R)=\int_{S_1}\frac{\partial M^*(\vec x,\vec\xi)}{\partial\vec n_\xi}dS_1(\vec x)=-\frac{dB(R)}{dR}=\frac{1-\nu}{2}R^2,$$

$$G(R)=-\int_{S_1}\frac{\partial\theta^*(\vec x,\vec\xi)}{\partial\vec n_\xi}dS_1(\vec x)=-\frac{dC(R)}{dR}=\frac{R}{2T},$$

$$H(R)=-\int_{S_1}\frac{\partial w^*(\vec x,\vec\xi)}{\partial\vec n_\xi}dS_1(\vec x)=-\frac{dD(R)}{dR}=-\frac{R}{2T}, \qquad (4)$$

where T – cylindrical stiffness of a plate, ν – Poisson's ratio. In the order of derivation of $A(R)$ it was used that

$$V^*(\vec x,\vec\xi)=Q^*(\vec x,\vec\xi)+\frac{\partial M_\tau^*(\vec x,\vec\xi)}{\partial S_1(\vec x)}dS_1(\vec x)=Q^*(\vec x,\vec\xi), \qquad (5)$$

where $Q^*(\vec x,\vec\xi)$ – shear force and $M_\tau^*(\vec x,\vec\xi)$ – torsion torque.

Using (3) and (4) it would be easy to conclude that the circular plate is in a state of pure bending.

Let us consider N-angle plate which is corresponds to using of the direct boundary elements method with the first order elements to a circular plate. In this case smooth boundary is replaced by set of N line segments Γ_j — boundary elements:

$$\Gamma \approx \bigcup_{j=1}^{N} \Gamma_j \,.$$

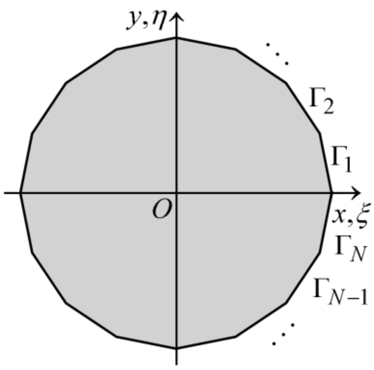

Fig. 2. The geometry of N -angle plate.

Therefore, it is hoped that

$$A(R) \approx \sum_{j=1}^{N} A_j \,, \qquad B(R) \approx \sum_{j=1}^{N} B_j \tag{6}$$

and so on. In (6) A_j and B_j represents integral from corresponding expression along Γ_j:

$$A_j = -\int_{-a_j}^{a_j} V^*(\vec{x},\vec{\xi})dy_j \,, \qquad B_j = -\int_{-a_j}^{a_j} M^*(\vec{x},\vec{\xi})dy_j \,, \tag{7}$$

where

$$V^*(\vec{x}_j,\vec{\xi}_j) = Q^*(\vec{x}_j,\vec{\xi}_j) + \frac{\partial M^*_{x_j y_j}(\vec{x}_j,\vec{\xi}_j)}{\partial y_j} \,,$$

$$M^*(\vec{x}_j,\vec{\xi}_j) = M^*_{x_j}(\vec{x}_j,\vec{\xi}_j), \tag{8}$$

where $\vec{x}_j$, $\vec{\xi}_j$ are coordinates of points $\vec{x}$ and $\vec{\xi}$ in the local coordinate system, which is associated with the *j*-th boundary element Γ_j (see Fig. 3).

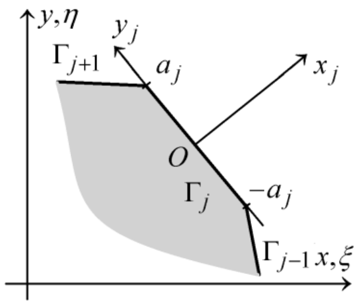

Fig. 3. Local coordinate system associated with Γ_j.

From second relationships in (7) and (8) we can derive

$$B_j(\vec{\xi}_j) = \int_{-a_j}^{a_j} M^*_{x_j}(\vec{x}_j,\vec{\xi}_j)dy_j =$$

$$= -T\int_{-a_j}^{a_j}\left(\frac{\partial w^*(\vec{x}_j,\vec{\xi}_j)}{\partial x_j^2} + \nu\frac{\partial w^*(\vec{x}_j,\vec{\xi}_j)}{\partial y_j^2}\right)dy_j =$$

$$= \frac{1}{8\pi}\Bigg[2(1-\nu)a_j +$$

$$+ 4\xi_j\left(\arctan\frac{\eta_j - a_j}{\xi_j} - \arctan\frac{\eta_j + a_j}{\xi_j}\right) +$$

$$+ (1-\nu)(\eta_j - a_j)\ln\left(\xi_j^2 + (\eta_j - a_j)^2\right) -$$

$$- (1-\nu)(\eta_j + a_j)\ln\left(\xi_j^2 + (\eta_j + a_j)^2\right).$$

It can be pronouncedly shown that

$$\lim_{N\to\infty}\sum_{j=1}^{N} B_j(\vec{\xi}_j) = \frac{1}{4}\left[(3+\nu) - (1-\nu)R^2\right] = B(R)$$

as well as

$$\lim_{N\to\infty}\sum_{j=1}^{N} C_j(\vec{\xi}_j) = C(R)\,, \quad \lim_{N\to\infty}\sum_{j=1}^{N} D_j(\vec{\xi}_j) = D(R)\,,$$

$$\lim_{N\to\infty}\sum_{j=1}^{N} E_j(\vec{\xi}_j) = E(R)\,, \quad \lim_{N\to\infty}\sum_{j=1}^{N} F_j(\vec{\xi}_j) = F(R)\,,$$

$$\lim_{N\to\infty}\sum_{j=1}^{N} G_j(\vec{\xi}_j) = G(R)\,, \quad \lim_{N\to\infty}\sum_{j=1}^{N} H_j(\vec{\xi}_j) = H(R)\,.$$

A completely different type of situation occurs with coefficient A. Accordantly to first relationships in (7) and (8) this coefficient can be divided on two parts:

$$A_j^{(1)}(\vec{\xi}_j) = -\int_{-a_j}^{a_j} Q^*(\vec{x}_j,\vec{\xi}_j)dy_j =$$

$$= \frac{1}{2\pi}\left[\arctan\frac{\eta_j - a_j}{\xi_j} - \arctan\frac{\eta_j + a_j}{\xi_j}\right] \tag{9}$$

and

$$A_j^{(2)}(\vec{\xi}_j) = -\int_{-a_j}^{a_j}\frac{\partial M^*(\vec{x}_j,\vec{\xi}_j)}{\partial y_j}dy_j =$$

$$= \frac{1-\nu}{4\pi}\left[\frac{\xi_j(\eta_j - a_j)}{\xi_j^2 + (\eta_j - a_j)^2} - \frac{\xi_j(\eta_j + a_j)}{\xi_j^2 + (\eta_j + a_j)^2}\right]. \tag{10}$$

It is worth mentioning that

$$\lim_{N\to\infty}\sum_{j=1}^{N} A_j^{(1)}(\vec{\xi}_j) = 1\,,$$

$$\lim_{N\to\infty}\sum_{j=1}^{N} A_j^{(2)}(\vec{\xi}_j) = \frac{1-\nu}{2}\left(1-R^2\right). \tag{11}$$

Consequently,

$$\lim_{N\to\infty}\sum_{j=1}^{N} A_j(\vec{\xi}_j) = .$$

$$= \lim_{N\to\infty}\sum_{j=1}^{N}\left[A_j^{(1)}(\vec{\xi}_j) + A_j^{(2)}(\vec{\xi}_j)\right] \neq A(R)\,. \tag{12}$$

Relation (12) is the heart of the matter of Sapondzyan's paradox.

Therefore, replacing the smooth contour by a polygon adds the component of the shear force, which is corresponding to the torque, and gives deviation in ending result. The comparative analysis of the expressions (10)–(12) with first expression (4) leads to the following conclusion: for implementing the numerical scheme of the direct boundary first-order elements method the shear force $Q^*(\vec{x},\vec{\xi})$ must be use instead of efficient shear force $V^*(\vec{x},\vec{\xi})$.

For verification of hypothesis, two problems were considered for the elliptical and for the square plates. The elliptical plate has the principal radiuses $r_x = 2$, $r_y = 1$ and the square plate has unit length of side. Poisson`s ratio of plates material is equal to 0.3. In the first problem the plates have been simply supported and constant value of bending moment $\overline{M} = T$ along boundary was specified. Also it was assumed that the deflection of plate along boundary is absent: $\overline{w} \equiv 0$. Second problem corresponds similar conditions, but deflection of plate along boundary was assumed equal constant nonzero value: $\overline{w} \equiv 1$.

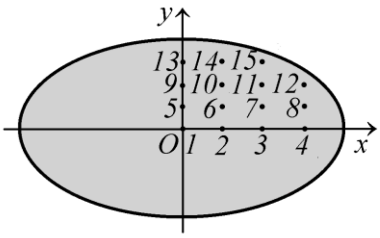

Fig. 4. Geometry and calculation points of elliptical plate.

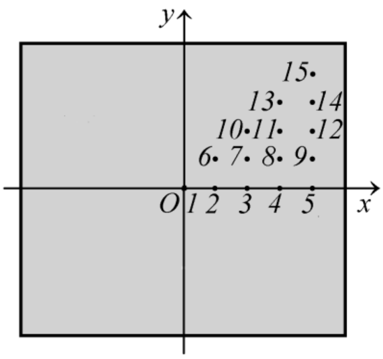

Fig. 5. Geometry and calculation points of elliptical plate.

Since the coefficient $V^*(\vec{x},\vec{\xi})$ is included in the equation as coefficient of the term of deflection, in the case where the deflection is zero at the boundary, it should not effect on the final result. But in the case when $\overline{w} \neq 0$ it matters whether we take for the corresponding coefficient term $V^*(\vec{x},\vec{\xi})$ or term $Q^*(\vec{x},\vec{\xi})$. The scheme of boundary element method with term $V^*(\vec{x},\vec{\xi})$ shall be called full scheme and the scheme with term $Q^*(\vec{x},\vec{\xi})$ shall be called short scheme.

For both scheme the results of first problem solving are identical to within 4 decimals. Such situation takes place for both plates. These solutions will be considering as exact solutions of problems.

Table 1. The relative deflection ($w(x,y) - \overline{w}$) of elliptical plate.

№	x	y	first problem ($\overline{w} = 0$)	second problem ($\overline{w} = 1$)	
				full scheme	short scheme
1	0.00	0.00	0.4973	0.4261	0.4973
2	0.50	0.00	0.4700	0.4057	0.4700
3	1.00	0.00	0.3860	0.3530	0.3860
4	1.50	0.00	0.2371	0.2757	0.2371
5	0.00	0.25	0.4657	0.3991	0.4657
6	0.50	0.25	0.4381	0.3772	0.4381
7	1.00	0.25	0.3531	0.3183	0.3531
8	1.50	0.25	0.2024	0.2231	0.2024
9	0.00	0.50	0.3713	0.3185	0.3714
10	0.50	0.50	0.3429	0.2935	0.3429
11	1.00	0.50	0.2553	0.2215	0.2553
12	1.50	0.50	0.0999	0.0952	0.0999
13	0.00	0.75	0.2154	0.1855	0.2154
14	0.50	0.75	0.1858	0.1583	0.1858
15	1.00	0.75	0.0947	0.0783	0.0947

Table 2. The relative deflection ($w(x,y) - \overline{w}$) of square plate

№	x	y	first problem ($\overline{w} = 0$)	second problem ($\overline{w} = 1$)	
				full scheme	short scheme
1	0.00	0.00	0.0738	0.1460	0.0738
2	0.10	0.00	0.0713	0.1412	0.0713
3	0.20	0.00	0.0635	0.1259	0.0635
4	0.30	0.00	0.0498	0.0975	0.0498
5	0.40	0.00	0.0291	0.0546	0.0291
6	0.10	0.10	0.0688	0.1370	0.0688
7	0.20	0.10	0.0614	0.1233	0.0614
8	0.30	0.10	0.0482	0.0969	0.0482
9	0.40	0.10	0.0283	0.0550	0.0283
10	0.20	0.20	0.0549	0.1143	0.0549
11	0.30	0.20	0.0434	0.0945	0.0434
12	0.40	0.20	0.0257	0.0568	0.0257
13	0.30	0.30	0.0347	0.0868	0.0347
14	0.40	0.30	0.0209	0.0612	0.0209
15	0.40	0.40	0.0131	0.0642	0.0131

Conclusions

1. Comparison the results of first and second problem solving, which are showed in the table 1 and table 2, lend support to the validity of the hypothesis that it necessity use shear force $Q^*(\vec{x},\vec{\xi})$ instead of shear force $V^*(\vec{x},\vec{\xi})$ in the numerical scheme of the direct boundary elements method of first-order.

2. From author's standpoint, the assumption based on the consideration of Sapondzyan's paradox had not been adequately explored in the literature on the method of boundary elements [2], [3]. Therefore, all of the above in this article may be a prerequisite for further discussion.

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ПРО ДЕЯКІ ОСОБЛИВОСТІ РЕАЛІЗАЦІЇ МЕТОДУ ГРАНИЧНИХ ЕЛЕМЕНТІВ У ТЕОРІЇ ЗГИНУ ПЛАСТИН

А. Г. Куценко, О. Г. Куценко, В. В. Яременко

Анотація. У роботі розглянута проблема, яка виникає при побудові чисельної схеми методу граничних елементів першого порядку в теорії пластин. При побудові такої схеми початково гладка границя пластини замінюється ламаною. При цьому виникає невідповідність чисельних результатів дійсному розподілу прогинів та інших характеристик. Причина даної невідповідності лежить в, так званому, парадоксі Сапонджяна. У відповідність до нього прогин пластини у вигляді правильного багатокутника не прагне до прогину круглої пластини при збільшенні кількості сторін багатокутника. У роботі вказані складові чисельної схеми методу граничних елементів, відповідальні за зазначену невідповідність, та представлена модифікація схеми, яка дозволяє усунути вказану проблему. Даний підхід був апробований на прикладі розв'язку двох пар задач для еліптичної і прямокутної пластин. Чисельні результати розв'язку задач показали адекватність запропонованої схеми.

Ключові слова: теорія пластин, прогин, метод граничних елементів, парадокс Сапонджяна

О НЕКОТОРЫХ ОСОБЕННОСТЯХ РЕАЛИЗАЦИИ МЕТОДА ГРАНИЧНЫХ ЭЛЕМЕНТОВ В ТЕОРИИ ИЗГИБА ПЛАСТИН

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Аннотация. В статье рассмотрена проблема, которая возникает при построении численной схемы метода граничных элементов первого порядка в теории пластин. При построении такой схемы изначально гладкая граница пластины заменяется ломаной. При этом возникает несоответствие численных результатов действительному распределению прогибов и прочих характеристик. Причина данного несоответствия лежит в так называемом парадоксе Сапонджяна. В соответствие с ним прогиб пластины в виде правильного многоугольника не стремиться к прогибу круглой пластины при увеличении количества сторон многоугольника. В работе указаны составляющие численной схемы метода граничных элементов, ответственные за отмеченное несоответствие, и представлена модификация схемы, позволяющая устранить данную проблему. Данный подход был апробирован на примере решения двух пар задач для эллиптической и прямоугольной пластин. Численные результаты решения задач показали адекватность предложенной схемы.

Ключевые слова: теория пластин, прогиб, метод граничных элементов, парадокс Сапонджяна

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