THE ESTIMATE FOR THE DT-MODULE OF SMOOTHNESS

O. Dyuzhenkova

In this article we consider the DT-module of smoothness, introduced by Ditzian and Totik. We investigate the connection between the DT-module of smoothness $\overline{\omega}_{k,r}(\tau, f^{(r)})$ of the r-s derivative of the function f and classical module of smoothness $\omega_k(\tau, \tilde{f}^{(r)})$ of the r-s derivative of the periodic function $\tilde{f} = f(\cos t)$. In particular, we get the upper estimate for the DT-module of smoothness.

An important chapter of approximation theory is constructive characteristics of classes of functions which are expressed through the evaluations of their uniform approximation by algebraic polynomials. Significant results to approximate the continuous on $[-1;1]$ functions were obtained in the terms of classical module of smoothness module of the order k for function $\tilde{f} = f(\cos t)$. Recently, for uniform approximation of functions the DT-modules of smoothness of Ditzian and Totik widely used. So to obtain the constructive characteristics of functions is important to explore the connection between the DT-module of smoothness and classical module of smoothness.

In researches we used the methods of uniform approximation of functions, in particular interpolation functions by algebraic polynomials and Lagrange polynomials.

Let $C^0_{[a;b]} := C_{[a;b]}$ – the space of continuous in $[a;b] \subset R$ functions $f:[a;b] \to R$ with uniform norm $\|f\|_{[a;b]} := \max\limits_{x \in [a;b]} |f(x)|$. Denote $C^r_{[a;b]} := \{ f \,|\, f^{(r)} \in C_{[a;b]} \}, r \in N$, and C^r – a subset of the functions $f \in C_{[-1;1]}$, that have continuous r-th derivative on the interval $(-1;1)$. Note $k \in N, h \in R$.

The module of smoothness of order k for function $f \in C_{[a;b]}$ is a function

$$\omega_k(\tau, f, [a;b]) := \sup_{h\in[0;\tau]} \left\|\Delta_h^k(f;x)\right\|_{[a+\frac{kh}{2};b-\frac{kh}{2}]}, \quad \tau \ge 0.$$

Marking $\varphi(x) := \sqrt{1-x^2}$ consider the modules of smoothness introduced by Z.Ditzian and Totik.

The DT-module of smoothness of order k for function $f \in C_{[-1;1]}$ is a function

$$\overline{\omega}_k(\tau, f) = \sup_{h\in[0;\tau]} \sup_{x:[x-\frac{kh\varphi(x)}{2};x+\frac{kh\varphi(x)}{2}]\subset[-1;1]} \left\|\Delta_{h\varphi(x)}^k(f;x)\right\|_{[a+\frac{kh}{2};b-\frac{kh}{2}]}, \quad \tau \ge 0$$

The DT-module of smoothness of order k with weight

$$\varphi_r := \varphi_r(x,k,h) := \left(1 + x - \frac{kh\varphi(x)}{2}\right)^{\frac{r}{2}} \left(1 - x - \frac{kh\varphi(x)}{2}\right)^{\frac{r}{2}}, \quad r \in R$$

for continuous in $(-1;1)$ function f is function

$$\overline{\omega}_{k,r}(\tau, f) := \sup_{h\in[0;\tau]} \sup_{x:[x-\frac{kh\varphi(x)}{2};x+\frac{kh\varphi(x)}{2}]\subset(-1;1)} \left\|\varphi_r \Delta_{h\varphi(x)}^k(f;x)\right\|_{[a+\frac{kh}{2};b-\frac{kh}{2}]}, \quad \tau \ge 0$$

To investigate the connection between the modules $\overline{\omega}_{k,r}(\tau, f^{(r)})$ and $\omega_k(\tau, \tilde{f}^{(r)})$ we estimate the DT-module of smoothness from above. As a result we get the theorem.

Theorem. For any $k \in N, r \in N$ and arbitrary function $f \in C_{[-1;1]}$ whose derivative $\tilde{f}^{(r)}(t)$ is continuous on the R follows inequality

$$\overline{\omega}_{k,r}(\tau, f^{(r)}) \le c_1 \omega_k(\tau, \tilde{f}^{(r)}), \quad \tau \ge 0.$$