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REFINED ALGORITHM FOR CALCULATING AREA DISTORTION ADJUSTMENTS OF GEOSPATIAL OBJECTS IN GEODETIC PROJECTIONS

S. Radov, Candidate of Technical Sciences, Associate Professor
E-mail: s.radov@chdtu.edu.ua
Cherkasy State Technological University
S. Rotte, Candidate of Technical Sciences, Associate Professor
E-mail: s.rotte@chdtu.edu.ua
Cherkasy State Technological University
A. Volontyr, Senior Lecturer
E-mail: a.volontyr@chdtu.edu.ua
Cherkasy State Technological University
V. Lytvyn, Assistant
E-mail: litvin_valyav@ukr.net
Cherkasy State Technological University
P. Stanko, Assistant
E-mail: p.stanko@chdtu.edu.ua
Cherkasy State Technological University

Abstract. *Area, as one of the fundamental geometric parameters of land parcels, is a key factor in land management, land cadaster, land taxation, and related fields. The geodetic area should ideally be determined on the surface of the reference ellipsoid; however, in engineering practice, it is usually calculated from the coordinates of turning points in the Gauss-Krüger projection. The influence of projection is approximately accounted for by introducing a correction to the computed area value derived from rectangular coordinates. At the same time, the geodetic area may differ significantly from its approximate value.*

There are no exact (closed-form) formulas to calculate the areas of arbitrary parcels on the ellipsoid surface based on the geodetic coordinates of turning points. Therefore, the most accurate values of geodetic areas of geospatial objects are obtained by numerical methods, including those used in geographic information *systems. A simpler method of determining “undistorted” land parcel areas relies on equal-area geodetic projections.*

This study employs the Sanson, Gauss-Krüger and Cassini projections. For the Gauss-Krüger and Cassini projections, algorithms are provided for calculating corrections based on the ordinate of the parcel’s centroid and the ordinates of all turning points. The accuracy of the proposed algorithms has been analyzed using mathematical modeling.

***Key words**: geodetic area, geodetic projections, area distortion, correction to calculated area.*

Problem Statement. Distortions of areas in cartographic projections have been known since ancient times [1]. One of the main methods for determining undistorted areas is the use of equal-area projections. However, in topographic and geodetic practice, conformal projections have demonstrated the highest efficiency. Among them, the transverse cylindrical Gauss-Krüger projection has been the principal system used in Ukraine since the early twentieth century [2].

To account for distortions of land parcel areas in this projection, the earliest recommendations involved the use of maps with known coefficients of distortion for elementary spheroidal trapezoids. Later, the practice shifted toward applying corrections to areas calculated from plane rectangular coordinates.

$$\Delta \boldsymbol{S} = \boldsymbol{S}_{x,y} \frac{\boldsymbol{y}^2}{\boldsymbol{R}^2}, \qquad (1)$$

where $S_{x,y}$ denotes the area of the plot in the Gauss-Kruger projection;

y is the ordinate of the area’s gravity center relative to the axial meridian;

R is the the average radius of the Earth’s ellipsoid curvature at the center of gravity.

The issue of area reduction in the Gauss-Krüger projection has not been addressed by textbooks on higher geodesy or mathematical cartography. It is generally assumed that distortions are practically negligible on sheets of topographic maps since they do not exceed other errors in determining areas on maps.

Review of Recent Research and Publications. The distortion of surface areas on the ellipsoid in any geodetic projection depends not only on the scale factors at the parcel's vertices but also on the curvature of the geodesic line in its planar representation. Under certain conditions, the influence of this factor may exceed the distortion due to the scale of representation *(*ΔS, eq. 1) as well as the root mean square error m_S of determining the parcel area, which is calculated according to the following formula:

$$\boldsymbol{m_S = m_P\sqrt{\frac{1}{8}\sum_{i=1}^{n} D_i^2}}, \qquad (2)$$

where m_P is the root mean square error of the position of boundary marks;

D_i is the length of the line between the previous (i-1) and next (i+1) characters.

Accounting for area distortions of land parcels is particularly relevant in light of current requirements for state land cadastre data. These requirements can be met through the development of methods and tools for geodetic measurements, the application of numerical methods for calculating geodetic areas of parcels, and the implementation of computer and geoinformation technologies [3.4].

For the first time in land management practice, the *Order* [2] introduced the requirement to use geodetic areas, which are recommended to be determined either by a rigorous method on the surface of the reference ellipsoid or by applying ΔS correction (eq. 1) to the area computed in the Gauss-Krüger projection. However, no criteria have been provided for selecting between the rigorous or the approximate method.

The aim of this study is to improve algorithms for determining geodetic areas based on plane rectangular coordinates and to substantiate criteria for the use of approximate and rigorous numerical methods.

Materials and Methods. Let us first study the equidistant transverse cylindrical projection of the ellipsoid onto the plane, in which geocentric coordinate ellipses are used instead of the large coordinate circles of the classical spherical Cassini projection [5]. In this ellipsoidal Cassini projection, when ordinates are undistorted (m_y=1), the abscissas of points are distorted in the same manner as in the Gauss-Krüger projection:

$$\boldsymbol{m_x = sec\omega}; \qquad (3)$$

$$\boldsymbol{\omega \approx \frac{y}{R}}, \qquad (4)$$

where ω denotes the angular argument of distortion;

y is the ordinate of a point with respect to the central meridian;

R=6378 km is the mean radius of curvature of the Earth ellipsoid over the territory of Ukraine [2].

We adopt the scale factor in the ordinate direction to construct a transformed equal-sized Cassini projection:

$$\boldsymbol{m_y = cos\omega}. \qquad (5)$$

Then the changed ordinate for the equal-sized projection will take the form:

$$\boldsymbol{y' \approx R \cdot sin\omega}, \qquad (6)$$

while ordinate increments from equal-magnitude to equal-intermediate projection can be calculated by the formula:

$$\Delta\boldsymbol{y = y - y' \approx y - R \cdot sin\frac{y}{R}}. \qquad (7)$$

Since the abscissas do not change when the scale of the ordinate distortion changes, the area correction can be calculated by the formula:

$$\Delta\boldsymbol{S = \frac{1}{2}\sum_{i=1}^{n}}\Delta\boldsymbol{y_i(x_{i-1} - x_{i+1})}, \qquad (8)$$

where i=1, 2, 3, …, n are the plots' turning point numbers.

It requires that the image scales of the m_x abscissa and the m_y ordinate be the same to build a transformed equiangular Cassini projection:

$$\boldsymbol{m_x = m_y = sec\omega}. \qquad (9)$$

Then the changed ordinate for an equiangular trapezoid takes the form:

$$\boldsymbol{y'' \approx R \cdot ln\, tg\left(\frac{\pi}{4} + \frac{\omega}{2}\right)}, \qquad (10)$$

whence angular argument of distortion is equal to

$$\boldsymbol{\omega = 2 \cdot \left\{arctg\left[\exp\left(\frac{y}{R}\right)\right] - \frac{\pi}{4}\right\}}, \qquad (11)$$

while ordinate increments from equal-area to equiangular projection are to be calculated by the formulas:

$$\Delta\boldsymbol{y'_i = R\omega_i - Rsin\omega_i}; \qquad (12)$$

$$\Delta\boldsymbol{y''_i = y_i - R\omega_i}; \qquad (13)$$

$$\Delta\boldsymbol{y_i} = \Delta\boldsymbol{y'_i} + \Delta\boldsymbol{y''_i = y_i - R \cdot sin\omega_i}. \qquad (14)$$

Area correction can be calculated by formula (8) or calculating solely the ordinate of the center of gravity y_m by the formula:

$$\Delta \boldsymbol{S} = \boldsymbol{S}_{\boldsymbol{x},\boldsymbol{y}} \cdot \boldsymbol{\omega}_{\boldsymbol{m}} \cdot \boldsymbol{sin\omega}_{\boldsymbol{m}}, \qquad (15)$$

where $S_{x,y}$ is the area of the plot in rectangular coordinates;

ω_m is the angular argument of distortion (11), calculated from the mean ordinate y_m;

Approximate value of the geodetic area is to be calculated by the formula:

$$\tilde{\boldsymbol{S}} = \boldsymbol{S}_{\boldsymbol{x},\boldsymbol{y}} - \Delta \boldsymbol{S}. \qquad (16)$$

Thus, to calculate the approximate value of the geodetic area from coordinates in conformal projections, we can use either the well-known correction formula (1) for the computed area, which is recommended for land management purposes [2], or the proposed formulas (15) and (8).

The necessity of accounting for the distortion of a land parcel's area can be assessed by comparing the calculated correction (1), (8), or (15) with the root mean square error of area determination (2). For asymmetric configurations with side lengths of 1-5 km, it is preferable to use equal-area projections, while for larger side lengths, geodetic areas should be determined numerically using approximated geodetic lines.

Results and Discussion. To analyze the accuracy of the proposed methods for accounting for area distortion, models of land parcels on the surface of the Earth's ellipsoid were created with both symmetric (circle, square, star, etc.) and arbitrary asymmetric contours. The geodetic coordinates of the turning points were determined by solving direct geodetic problems using Bessel's formulas, and the geodetic areas of the parcels were computed along approximated geodetic lines on the ellipsoid surface. Subsequently, using the corresponding plane coordinates in the Sanson, Cassini, and Gauss-Krüger projections, the areas, distortion corrections, and adjusted area values were calculated.

Mathematical modeling confirmed the applicability of all the methods considered under certain conditions. Among them, the correction according to formula (8) provided the highest accuracy, while formula (1) yielded the lowest.

For example, the final area errors after accounting for scale distortions in the Gauss-Krüger and Cassini projections for a symmetric model (Figure 1), with a geodetic area of S = 15,307.3 m², are presented in Table 1. The central point of the model is located at latitude B = 50° and longitudes from L = 0° to L = 40°, with distances from the central point to the respective points of 100 and 50 meters. The root mean square error (RMSE) of the area determined by Gauss-Krüger coordinates is 0.9 m² (5.9E-3 %) for a positional RMSE of 1 cm for the defining points.

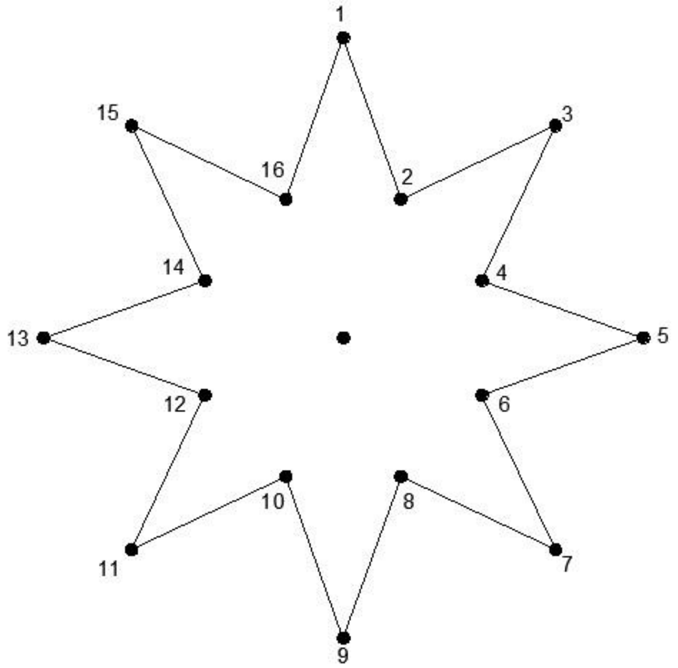

Figure 1. Schematic land plot model

Table 1. Area distortion calculation results

L°	Gauss-Krüger projection			Cassini projection		
	δS, m^2	δS_1, m^2	δS_2, m^2	δS, m^2	δS_2, m^2	δS_3, m^2
0°	0.0	0.0	0.0	0.0	0.0	0.0
10°	193.6	-1.9	-0.3	96.6	0.3	0.0
20°	779.4	-27.4	-12	385.5	4.7	0.0
30°	1767.0	-136.4	-3.4	861.3	23.0	0.1
40°	3155.5	-424.3	-7.7	1508.7	68.0	0.3

In the presented table, the following notations are adopted: δS is the distortion of the area calculated from the rectangular coordinates of points in the corresponding projection; δS_1 is the final area distortion after applying the correction calculated by formula (1); δS_2 is the final distortion after applying correction (15); δS_3 is the final distortion after applying correction (8).

Conclusions and Recommendations. The values of land parcel areas calculated from Gauss-Krüger plane rectangular coordinates are always larger than the corresponding geodetic areas due to the effect of the scale of area representation in this projection. Therefore, to account for area distortions, corrections (1), (8), or (15) must be applied to the computed area whenever the correction value exceeds the root mean square error (RMSE) of area determination (2). In addition, area distortions are also affected by the curvature of the mapped geodetic lines. To account for this factor, it is preferable to use numerical methods of area determination based on approximated geodetic lines. For line lengths of up to 5 km within a three-degree coordinate zone, equal-area projections can be used to reliably determine geodetic areas.

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С. Радов, С. Ротте, А. Волонтир? П. Станько, В. Литвин

УДОСКОНАЛЕННЯ АЛГОРИТМУ ВИЗНАЧЕННЯ ПОПРАВКИ ЗА СПОТВОРЕННЯ ПЛОЩІ ГЕОПРОСТОРОВИХ ОБ'ЄКТІВ В ГЕОДЕЗИЧНИХ ПРОЄКЦІЯХ

Анотація. *Площа, як один з основних геометричних параметрів земельних ділянок, є важливим фактором у землеустрої, земельному кадастрі, стягненні плати за землю тощо. Геодезична площа має визначатися на поверхні референц-еліпсоїда, але в інженерній практиці площа обчислюється за координатами поворотних точок в проєкції Гаусса-Крюгера. Вплив проєкції наближено враховується введенням поправки до обчисленого значення площі за прямокутними координатами. При цьому геодезична площа може суттєво відрізнятися від її наближеного значення.*

Не існує точних (кінцевих) формул для обчислення площ довільних ділянок поверхні еліпсоїду за геодезичними координатами поворотних точок. Тому найбільш точні значення геодезичних площ геопросторових об'єктів визначаються числовими методами, в тому числі у геоінформаційних системах. Простіший спосіб визначення неспотворених площ земельних ділянок полягає у використанні рівновеликих геодезичних проєкцій.

В дослідженні використані проєкції Сансона, Гаусса-Крюгера та Кассіні. Для проєкцій Гаусса-Крюгера та Кассіні наведені алгоритми визначення поправок за ординатою центра тяжіння ділянки та за ординатами усіх поворотних точок. Аналіз точності запропонованих алгоритмів проведений з використанням методу математичного моделювання.

***Ключові слова**: геодезична площа, геодезичні проєкції, спотворення площ, поправка до обчисленої площі.*